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USING THE FINITE-ELEMENT APPROACH FOR CONTOURING OF IRREGULARLY SPACED DATA[1]

CHUEN-TIEN SHYU[2]

ABSTRACT

If a contour map can reveal local anomalies both in their positions and values correctly, in general, it is considered of having the priority over the others. Usually, data points to be contoured are irregularly spaced; contour maps are necessary to be updated. These are overcome by solving the minimum potential equation with the finite-element method. Complete cubic triangular elements are chosen for the formulation. The corresponding matrix equation is then derived. As an illustration, the magnetic anomalies in the eastern Taiwan offshore area are contoured by the proposed method. The results appear quite promise.

INTRODUCTION

Over the past 30 years, the geophysical instruments have been made a significantly improvement on the acquisition of data. The amount of work of doing contour maps for data interpretation is, therefore, parallelly increased. With the help of high-speed digital computers, the automatic machine contouring has become a very attractive alternative of producing contour maps.

In most cases, the data points are not regularly spaced, e. g., along ship's track lines. A great variety of techniques have been used to calculate the point values in the vicinity of the recorded data. These methods are generally using weighting or interpolation functions or both to estimate the unknowns. Examples of machine contouring studies incorporating the above approachs can be found on Crain and Bhattacharyya (1967), Crain (1970), Falconer (1971), Bhattacharyya and Ross (1972). Slootweg (1978) has further applied the weighted mean method with a spatial filter to bathymetric data and concluded that his method is useful to minimize the influence of data distribution effects and positioning errors on non-analytic (e. g., bathymetric survey) as well as analytic (e. g., gravity or magnetic survey) data. Similarly, the interpolation of potential (analytic) field on the other hands, is a common problem in geophysical data processing. Borrowing the concept of spline interpolation, Briggs (1974) introduces the algorithm of solving partial differential equation for the bending plate. The derived finite difference equations are based on the principle of minimum total curvature. A FORTRAN IV program with examples is also available (Swain, 1976). With this approach, in general, will produce the smoothest surface passing through the observed values commonly produced by reconnaissance surveys. However, the main drawbacks of their approachs are,

1. the contouring lines may not pass exactly through the data values;
2. some interesting points (e. g., sudden peaks or troughs) may be either overlooked or biased due to the awkward of grid spacing;
3. need to reconstruct a data file for updating a contour map when more data become available in the area.

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1. Contribution No. 175 Institute of Oceanography, National Taiwan University, Taipei, Taiwan, Rep. of Chna.
2. Institute of Oceanography, National Taiwan University.

The present paper described an alternative approach, the finite-element method, of solving the biharmonic equation (i. e., the differential equation of a bending plate) with boundary values.

THE METHOD

Biharmonic Equation

The theory of thin plate flexure is developed in detail in many texts (e. g., Timoshenko and Woinowsky-Krieger, 1969; Marguerre and Woernle, 1969) and is outlined here simply to furnish the necessary equations for subsequent illustrations of element formulation. For an isotropic non-uniformly forced thin plate, the displacement u will satisfy

$$\nabla^2 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f \qquad \text{in } \Omega, \tag{1}$$

where $f(x, y)$ represents the bending force. As usual, the required boundary conditions will be half the order of the equation, so that for the present case, on some part of the boundary (external and internal), the value of u may be specified such as

$$u = u(x, y),$$

while on the remaining part of the boundary we have the condition corresponding to a free boundary; it is often written as

$$\frac{\partial^2 u}{\partial n^2} + \nu\left(\frac{\partial \varphi}{\partial s}\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial s^2}\right) = 0,$$

where n and s are the directions of the outward normal and tangent respectively, ν is the Poisson's ratio, and φ is the angle between the normal and the x-axis.

Finite-Element Method

The biharmonic equation (1) together with all the above boundary conditions can be coped with somehow by directly replacing derivatives by finite differences. The construction of accurate equations near the boundary becomes fantastically complicated, moreover, an alternative of much better starting point can be derived by taking $\nu = 1$ on the boundary in the variatioual form of (1). Then, the problem turns out to minimize the total potential energy

$$W(u) = \iint_{\Omega}\left\{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2\right]\right\} dxdy \tag{2}$$

subject to the appropriate boundary conditions such as the one proposed by Briggs (1974). For the bending force $f(x, y)$, the specified $u(x, y)$ is imposed. At the free boundary, there are no constraints and the space of admissible u is hold.

It is mathematically apparent that the functional $W(u)$ involves second derivatives of u and therefore demands continuity of the first derivatives. This requirement for two-dimensional elements can not be satisfied in any simple way. Though, the number of degree of freedom for a triangular element may be expanded until a coincidence is obtained with a complete polynomial that also satisfies the interelement displacement-continuity condition; this requires a quintic polynomial of 21 degrees of freedom. More degrees of freedom, of course, needs more computational efforts. When a great number of subregions are required to divide a spatial domain, like the one we are dealing with, the rigorous quintic element leads to awkward computational strategies. Weighing the flexibility and computational expense against the accuracy (Abel and Desai, 1972) a complete cubic triangular element (10 degrees of freedom) becomes a reasonable choice for the problem to be investigated. This is so-called incompatible

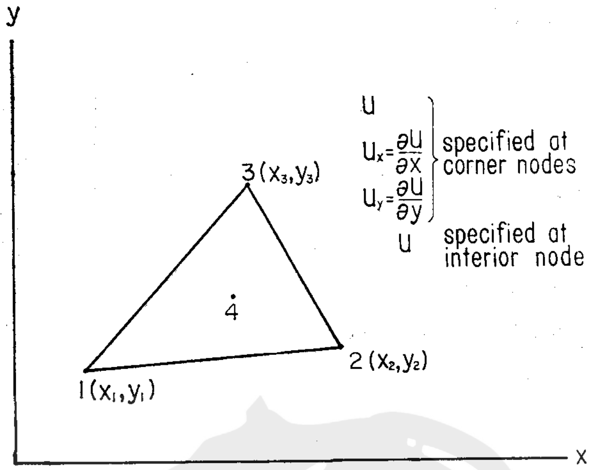

Fig. 1. Triangular element with 10 degrees of freedom.

or nonconforming elements. Incompatibe elements have often been used with surprising success in plate-bending problems (Holland, 1969). The element is described in Fig. 1. The value of u, $\partial u/\partial x$, and $\partial u/\partial y$ at the three vertices and u alone at the centroid is defined. Thus the u is related by

$$u = [N]\{\bar{u}\} \tag{3}$$

where the entries in $[N]$, in terms of triangular coordinates, are

$$\begin{aligned}
N_1 &= L_i^2(3-2L_i) - 7L_iL_jL_k \\
N_2 &= L_i^2(x_{ji}L_j - x_{ik}L_k) + (x_{ik} - x_{ji})L_iL_jL_k \\
N_3 &= L_i^2(y_{ji}L_j - y_{ik}L_k) + (y_{ik} - y_{ji})L_iL_jL_k \\
&\vdots \\
N_{10} &= 27L_1L_2L_3 \\
x_{ij} &= x_i - x_j \text{ and } y_{ij} = y_i - y_j
\end{aligned} \tag{4}$$

with (i, j, k) cyclic permutations of (1, 2, 3).

Suppose that the solution domain Ω is divided into m elements of 10 degrees of freedom each, the functional $W(u)$ can be represented as the sum of W_e overall the elements, that is,

$$W(u) = \sum_{e=1}^{m} W_e(u)$$

The discretized form of the potential (2) for one element is obtained by substituting equation (3) into equation (2). Then the minimum condition for one element becomes

$$\frac{\partial W_e(\bar{u})}{\partial \bar{u}_i} = 0, \qquad i = 1, 2, \cdots, 10 \tag{5}$$

Following the standard procedure to evaluate each of the derivatives we have

$$[K]\{\bar{u}\} = \{R\} \tag{6}$$

where

$$K_{ij}=\iint_{\Omega^{(e)}} N_i'' N_j'' \{\bar{u}\} dxdy, \qquad i=1, 2, \cdots, 10; \quad j=1, 2, \cdots, 10$$

$$N_i''=\frac{1}{\det[J]}\begin{bmatrix} y_{23}^2 & -2y_{13}y_{23} & y_{13}^2 \\ -x_{23}y_{23} & x_{13}y_{23}+y_{13}x_{23} & -x_{13}y_{13} \\ x_{23}^2 & -2x_{13}x_{23} & x_{13}^2 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial^2 N_i}{\partial L_1^2} \\ \dfrac{\partial^2 N_i}{\partial L_1 \partial L_2} \\ \dfrac{\partial^2 N_i}{\partial L_2^2} \end{Bmatrix}$$

$$\det[J]=(x_{13}y_{23}-y_{13}x_{23})^2$$

$$R_i=-\sum_{k=1} K_{ij}u_k \text{ for specified } u_k,$$
$$=0 \text{ otherwise.}$$

TEST EXAMPLE

As an illustration of how the technique actually inerpolates the values from the existent data, we present a sample problem as follows. Consider a flat plate subjected to specified displacements along the boundary. Fig. 2 illustrates qualitatively the solutions of the displacement field. Even with a relatively coarse-element mesh (32 elements are used) the interpolated values are reasonable.

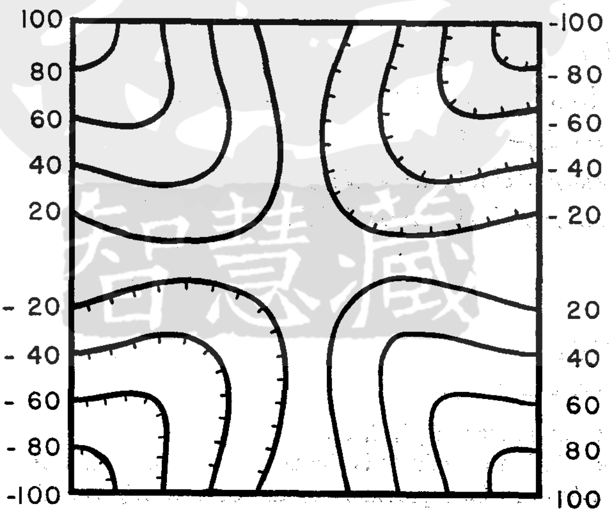

Fig. 2. Calculated unknowns from the given boundary values.

APPLICATIONS

The field data at hand can be used to illustrate the method are the marine magnetic anomalies collected in 1973, 1975, and 1979 by the Institute of Oceanography, National Taiwan University (Fig. 3).

For the contouring we have divided the area into 800 elements and solved up to about 2000 equations. The area contoured in Fig. 4 is in the north-eastern Taiwan offshore—a complicate junction of the Ryukyu arc and the Luzon arc. The negative anomalies in the south and the positive in the north-east respectively are considered as a result of the subduction of

CONCLUSIONS

To incorporate with the finite-element method into the equation of minimum potential has made the sampling of irregularly spaced data for machine contouring without difficulty. Data obtained at different time may be compiled and renewed by simply insert additional elements; The old prepared files for the contouring of the previous maps, hence, need not to be destroyed. The only disadvantage of the present method in comparison with the finite difference is the need of solving more equations.

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利用有限元素法描繪不等間隔資料的等值線圖

徐 春 田

摘 要

一張等值線圖最重要的是要能正確的反映出局部性異常的值及位置。通常所得的資料多是以任意形狀分佈的，而且將來如果該區域的資料增多時有隨時重繪等值線圖的可能。爲解決這些問題，我們用有限元素法去解最少總位能的積分方程式，選擇最合理的三次三角形的單位元素代入，由此導出矩陣方程式。爲了說明這種方法的效用，我們處理了臺灣東部沿海的磁力異常值，結果良好，對將來處理資料繪製等值圖技巧的改進頗具遠景。